A bound on 6D $\mathcal{N} = 1$ supergravities

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ABSTRACT: We prove that there are only finitely many distinct semi-simple gauge groups and matter representations possible in consistent 6D chiral $(1,0)$ supergravity theories with one tensor multiplet. The proof relies only on features of the low-energy theory; the consistency conditions we impose are that anomalies should be cancelled by the Green-Schwarz mechanism, and that the kinetic terms for all fields should be positive in some region of moduli space. This result does not apply to the case of the non-chiral $(1,1)$ supergravities, which are not constrained by anomaly cancellation.

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Contents

1 Introduction

The requirement of anomaly cancellation can provide non-trivial consistency conditions for quantum field theories. In the context of the Standard Model, for example, vanishing of gauge anomalies requires that the number of generations of leptons and quarks are equal. The anomalies in any given model are completely determined by the low-energy effective theory, and in particular, by massless matter fields that violate parity (chiral fermions, self-dual tensors, etc). Anomalies arise at one loop and receive no further quantum corrections; this makes them easy to compute given the matter content of the model (for a review, see [1]). When the dimension of space-time is of the form $4k+2$, there are purely gravitional, purely gauge and mixed gravitional-gauge anomalies [2]. In the case of ten dimensional $\mathcal{N} = 1$ (chiral) supergravity coupled to a vector multiplet, it was shown by Green and Schwarz [3] that anomalies can be cancelled only when the gauge group is $SO(32)$, $E_8 \times E_8$, $U(1)^{248} \times E_8$ or $U(1)^{496}$. This is a powerful constraint on this class of theories. Our goal here is to explore the analogous constraint in six-dimensional supergravities.

In this paper, we consider the constraints from anomaly cancellation in the case of six-dimensional $(1, 0)$ supergravity coupled to one tensor multiplet, and any number of vector multiplets and matter hypermultiplets. We prove the following result:

The set of semi-simple gauge groups and matter representations appearing in consistent chiral, six-dimensional, $(1, 0)$ supergravity models which have positive gauge-kinetic terms and one tensor multiplet is finite.

We restrict to the case of one tensor multiplet because in this case the details of the anomaly cancellation are simpler, and the theory has a Lagrangian description . We have also assumed that the gauge group contains no U(1) factors. We expect that a generalization of the proof given in this paper to include abelian gauge group factors and an arbitrary number of tensor multiplets may be possible. Since we restrict ourselves to *chiral* $(1,0)$ supergravity theories, our conclusions do not apply to models with $(1,1)$ supergravity, which are always anomaly free.

In section 2, we review some properties of the models we consider and discuss the constraints from anomaly cancellation. In section 3, we prove that the set of models is finite. In section 4, we discuss the possibility of a systematic classification of allowed models. In section 5, we summarize and conclude with some comments on future directions.

2 Six-dimensional (1,0) supergravity and anomaly cancellation

We are interested in six-dimensional theories with $(1,0)$ supersymmetry which are gauge theories with matter coupled to gravity and a single tensor multiplet. The field content of such a theory consists of single $(1,0)$ gravity and tensor multiplets, n_v vector multiplets, and n_h hyper multiplets. Table 1 summarizes the matter content of the $(1,0)$ supersymmetry multiplets we consider in this paper. We assume that the gauge group $\mathcal{G} = \prod_i G_i$ is semi-simple and contains no U(1) factors. The low-energy Lagrangian for such a model takes the form (see [4–7])

$$\mathcal{L} = -\frac{1}{2}e^{-2\phi}(dB - \omega) \cdot (dB - \omega) - B \wedge d\tilde{\omega} - \sum_i (\alpha_i e^\phi + \tilde{\alpha}_i e^{-\phi})\mathrm{tr}(F_i^2) + \cdots \qquad (2.1)$$

The index i runs over the various factors in the gauge group. ϕ is the scalar in the tensor multiplet. Note that there are many terms in the Lagrangian, including the hypermultiplet kinetic terms, gravitational couplings and fermion terms, which we have not included as they are not relevant for our discussion (for details see [4, 5]). $\omega, \tilde{\omega}$ are Chern-Simons forms defined as

$$d\omega = \frac{1}{16\pi^2}\left(\mathrm{tr}R \wedge R - \sum_i \alpha_i \ \mathrm{tr}F_i \wedge F_i\right) \qquad (2.2)$$

$$d\tilde{\omega} = \frac{1}{16\pi^2}\left(\mathrm{tr}R \wedge R - \sum_i \tilde{\alpha}_i \ \mathrm{tr}F_i \wedge F_i\right) \qquad (2.3)$$

The B-field transforms under a gauge transformation Λ_i in the factor G_i as

$$\delta B = -\frac{1}{16\pi^2}\sum_i \alpha_i \mathrm{tr}(\Lambda_i F_i) \qquad (2.4)$$

The Lagrangian (2.1) is not gauge invariant, because of the $B \wedge d\tilde{\omega}$ coupling; this term is needed for the six-dimensional analogue [8] of the ten-dimensional Green-Schwarz mechanism[1] [3]. The tree-level gauge variation of the Lagrangian cancels the one-loop quantum

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[1]In the case of six-dimensional models with multiple tensor multiplets, there is a generalized mechanism due to Sagnotti [9, 10]. Since we only consider the case of one tensor multiplet here, this mechanism is not relevant to the analysis of this paper.

Multiplet	Matter Content
SUGRA	$(g_{\mu\nu}, B^-_{\mu\nu}, \psi^-_\mu)$
Tensor	$(B^+_{\mu\nu}, \phi, \chi^+)$
Vector	(A_μ, λ^-)
Hyper	$(4\varphi, \psi^+)$

Table 1. Representations of $(1,0)$ supersymmetry in 6D. The $+$ and $-$ indicate the chirality for fermions and self-duality or anti-self-duality for the two index tensor.

anomaly, and as a result, the full quantum effective action is anomaly free. We discuss only some aspects of this mechanism here, referring the reader to [11–13] for further details.

The gauge anomaly in a d-dimensional theory is related to the chiral anomaly in $d+2$ dimensions, where it is expressed as a $d+2$-form [1]. Given a six-dimensional $(1,0)$ model with one tensor multiplet, the anomaly polynomial is an 8-form and takes the form [14]

$$I = -\frac{n_h - n_v - 244}{5760}\mathrm{tr}R^4 - \frac{44 + n_h - n_v}{4608}(\mathrm{tr}R^2)^2 - \frac{1}{96}\mathrm{tr}R^2 \sum_i \left[\mathrm{Tr}F_i^2 - \sum_R x_R^i \mathrm{tr}_R F_i^2 \right]$$
$$+ \frac{1}{24} \left[\mathrm{Tr}F_i^4 - \sum_R x_R^i \mathrm{tr}_R F_i^4 \right] - \frac{1}{4} \sum_{i,j,R,S} x_{RS}^{ij}(\mathrm{tr}_R F_i^2)(\mathrm{tr}_S F_j^2), \tag{2.5}$$

where F_i denotes the field strength for the simple gauge group factor G_i, tr_R denotes the trace in representation R of the corresponding gauge group factor, and Tr denotes the trace in the adjoint representation. x_R^i and x_{RS}^{ij} denote the numbers of hypermultiplets in representation R of G_i, and (R, S) of $G_i \times G_j$. We can express the traces in terms of the trace in the fundamental representation.

$$\mathrm{tr}_R F^2 = A_R \mathrm{tr}F^2$$
$$\mathrm{tr}_R F^4 = B_R \mathrm{tr}F^4 + C_R(\mathrm{tr}F^2)^2 \tag{2.6}$$

Note that tr (without any subscript) will always denote the trace in the fundamental representation. Formulas for the group-theoretic coefficients A_R, B_R, C_R can be found in [14].[2] The polynomial I can be written in terms of $\mathrm{tr}R^4, \mathrm{tr}F_i^4, \mathrm{tr}R^2, \mathrm{tr}F_i^2$ using (2.6). We rescale the polynomial so that the coefficient of the $(\mathrm{tr}R^2)^2$ term is one. Anomalies can be cancelled through the Green-Schwarz mechanism when this polynomial can be factorized as

$$I = \left(\mathrm{tr}R^2 - \sum_i \alpha_i \mathrm{tr}F_i^2 \right) \left(\mathrm{tr}R^2 - \sum_i \tilde{\alpha}_i \mathrm{tr}F_i^2 \right) \tag{2.7}$$

A necessary condition for the anomaly to factorize in this fashion is the absence of any irreducible $\mathrm{tr}R^4$ and $\mathrm{tr}F_i^4$ terms. This gives the conditions

$$\mathrm{tr}R^4 : \quad n_h - n_v = 244 \tag{2.8}$$

$$\mathrm{tr}F_i^4 : \quad B_{Adj}^i = \sum_R x_R^i B_R^i \tag{2.9}$$

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[2]Note that in [14], the coefficients v, t, u correspond to A_R, B_R, C_R respectively.

For groups G_i which do not have an irreducible $\mathrm{tr}F_i^4$ term, $B_R^i = 0$ for all representations R and therefore (2.9) is always satisfied. The sum in (2.9) is over all hypermultiplets that transform under any representation R of G_i. For example, a single hypermultiplet that transforms in the representation (R, S, T) of $G_i \times G_j \times G_k$ contributes $\dim(S) \times \dim(T)$ to x_R^i.

The condition (2.8) plays a key role in controlling the range of possible anomaly-free theories in 6D. The anomaly cancellation conditions constrain the matter transforming under each gauge group so that the quantity $n_h - n_v$ in general receives a positive contribution from each gauge group and associated matter, and the construction of models compatible with (2.8) thus has the flavor of a partition problem. While there are some exceptions to this general rule, and some subtleties in this story when fields are charged under more than one gauge group, this idea underlies several aspects of the proof of finiteness given here; we discuss this further in section 4.

After rescaling the anomaly polynomial so that the coefficient of $(\mathrm{tr}R^2)^2$ is one, when (2.8) is satisfied (2.5) becomes

$$I = (\mathrm{tr}R^2)^2 + \frac{1}{6}\mathrm{tr}R^2 \sum_i \left[\mathrm{Tr}F_i^2 - \sum_R x_R^i \mathrm{tr}_R F_i^2 \right] - \frac{2}{3} \left[\mathrm{Tr}F_i^4 - \sum_R x_R^i \mathrm{tr}_R F_i^4 \right]$$
$$+ 4 \sum_{i,j,R,S} x_{RS}^{ij} (\mathrm{tr}_R F_i^2)(\mathrm{tr}_S F_j^2) \tag{2.10}$$

For a factorization to exist, in addition to (2.8) and (2.9), the following equations must have a solution for real $\alpha_i, \tilde{\alpha}_i$

$$\alpha_i + \tilde{\alpha}_i = \frac{1}{6} \left(\sum_R x_R^i A_R^i - A_{Adj}^i \right) \tag{2.11}$$

$$\alpha_i \tilde{\alpha}_i = \frac{2}{3} \left(\sum_R x_R^i C_R^i - C_{Adj}^i \right) \tag{2.12}$$

$$\alpha_i \tilde{\alpha}_j + \alpha_j \tilde{\alpha}_i = 4 \sum_{R,S} x_{RS}^{ij} A_R^i A_S^j \tag{2.13}$$

Notice that the coefficients in the factorized anomaly polynomial $\alpha_i, \tilde{\alpha}_i$, are related to the coefficients in the $*B \cdot (F_i \wedge F_i)$ and $B \wedge F_i \wedge F_i$ terms in the Lagrangian (2.1). This was first observed in [9], and its consequences were discussed further in [6, 7, 10]. The coefficients $\alpha_i, \tilde{\alpha}_i$ are fixed by the anomaly polynomial, which in turn is determined completely by the choice of the gauge group and the matter content. Moreover, due to their origin in the anomaly, the coefficients $\alpha_i, \tilde{\alpha}_i$ are immune to quantum corrections. The coefficients $\alpha_i, \tilde{\alpha}_i$ play a key role in the structure of consistent 6D supergravity theories. In particular, the signs of these terms affect the behavior of the gauge fields in the theory at different values of the dilaton through the gauge kinetic terms in (2.1) [9]. When the gauge kinetic term controlled by $\alpha_i e^\phi + \tilde{\alpha}_i e^{-\phi}$ becomes negative, the theory develops a perturbative instability. When the gauge kinetic term vanishes, the gauge field becomes strongly coupled and the theory flows to an interacting superconformal field theory in the IR, with tensionless string excitations [7]. While there are potentially interesting features

associated with some of these superconformal theories [15, 16], in this work we focus on theories where all gauge kinetic terms are positive for some value of the dilaton. This imposes the constraint that there exists some ϕ so that for each gauge group at least one of $\alpha_i, \tilde{\alpha}_i$ is positive.

As stated in the introduction, the goal of this paper is to demonstrate that only a *finite* number of gauge groups and matter representations are possible in consistent 6D chiral supergravity theories with one tensor multiplet. We prove this in the following section, based only on the following two assumptions regarding the theory:

1. The anomalies can be cancelled by the Green-Schwarz mechanism.

2. The kinetic terms for all fields are positive in some region of moduli space.   (2.14)

Note in particular that the statement and proof of the finiteness of this class of theories is purely a statement about the low-energy effective theory, independent of any explicit realization of a UV completion of the theories, such as string theory. Clearly, it is of interest to understand which low-energy theories admit a consistent UV completion as quantum gravity theories. In [17], we conjectured that all UV-consistent 6D supergravity theories can be realized in some limit of string theory. The constraints on 6D supergravity theories arising from anomaly cancellation have intriguing structural similarity to the constraints associated with specific string compactification mechanisms [18–20]. This correspondence was made explicit for a particular class of heterotic compactifications in [17], and has been explored in the context of F-theory by Grassi and Morrison [21, 22], who showed that the anomaly cancellation conditions in 6D give rise to new nontrivial geometric constraints on Calabi-Yau compactifications. The proof of finiteness in this paper suggests a systematic approach to classifying the complete set of consistent chiral 6D supergravity theories. As we discuss in section 4, such a classification would help in making a more precise dictionary between the structure of low-energy theories and string compactifications.

3   Finiteness of anomaly-free models in 6D

We now proceed to prove the finiteness result stated in the introduction, using constraints 1 and 2 above. We will prove finiteness by contradiction. Assume there exists an infinite family $\mathcal{F}$ of distinct models $\mathcal{M}_1, \mathcal{M}_2, \ldots$. Each model $\mathcal{M}_\gamma$ has a nonabelian gauge group $\mathcal{G}_\gamma$ which is a product of simple group factors, and matter hypermultiplets that transform in arbitrary representations of the gauge group. The number of hypermultiplets, the matter representations and the gauge groups themselves are constrained by anomaly cancellation. We show that these constraints are sufficient to demonstrate that no infinite family of anomaly-free models exists, with distinct combinations of gauge groups and matter.

We first examine some simple consequences of the anomaly constraints. The absence of purely gravitational anomalies requires that $n_h - n_v + 29n_t = 273$, and for the case of one tensor multiplet (2.8) $n_h - n_v = 244$. If every model in an infinite family $\mathcal{F}$ has the same gauge group $\mathcal{G}_\gamma = \mathcal{G}$, but with distinct matter representations, we immediately arrive at a contradiction. If $\{R_i\}$ denotes the irreducible representations of $\mathcal{G}$, then we must

have $\sum_i x_i \dim(R_i) = \dim(\mathcal{G}) + 244$. Since $x_i \geq 0\ \forall i$, and the number of representations of $\mathcal{G}$ below a certain dimension is bounded, there are only a finite number of solutions to (2.8). Therefore, there exists no such infinite family. If we assume that model $\mathcal{M}_\gamma$ has a gauge group $\mathcal{G}_\gamma$, such that the dimension of the gauge group is always bounded from above $(\dim(\mathcal{G}_\gamma) < M,\ \forall \gamma)$, we again arrive at a contradiction. This is because there are only finitely many gauge groups below a certain dimension, and again the gravitational anomaly condition only has a finite number of solutions.

For any infinite family of anomaly-free models, the dimension of the gauge group is therefore necessarily unbounded. There are two possibilities for how this can occur, and we show that in each case the assumption of the existence of an infinite family leads to a contradiction.

1. The dimension of each simple factor in $\mathcal{G}_\gamma$ is bounded. In this case, the number of simple factors is unbounded over the family.

2. The dimension of at least one simple factor in $\mathcal{G}_\gamma$ is unbounded. For example, the gauge group is of the form $\mathcal{G}_\gamma = \mathrm{SU}(N) \times \tilde{G}_\gamma$, where $N \to \infty$.

Case 1: Bounded simple group factors. Assume that the dimension of each simple group factor is bounded from above by M. In this case, the number of factors in the gauge group necessarily diverges for any infinite family of distinct models $\{\mathcal{M}_\gamma\}$. Any model in the family has a gauge group of the form $\mathcal{G}_\gamma = \prod_{k=1}^{N} G_k$, where each G_k is simple with $\dim(G_k) \leq M$ and N is unbounded over the family.

Since each model in the family is free of anomalies, the anomaly polynomial for each model factorizes as

$$I = \left(\mathrm{tr}R^2 - \sum_{k=1}^{N} \alpha_k \mathrm{tr}F_k^2 \right) \left(\mathrm{tr}R^2 - \sum_{k=1}^{N} \tilde{\alpha}_k \mathrm{tr}F^2 \right) \tag{3.1}$$

The positivity condition on the kinetic terms (2.14) requires that at least one of $\alpha_k, \tilde{\alpha}_k$ is positive, for each factor G_k in the gauge group. The coefficient of $\mathrm{tr}F_i^2 \mathrm{tr}F_j^2$ in the anomaly polynomial (2.5) is related to the number of hypermultiplets charged simultaneously under both G_i and G_j. Hence, for every pair of factors G_i and G_j in the gauge group, the corresponding coefficients in the anomaly polynomial must satisfy

$$\alpha_i \tilde{\alpha}_j + \alpha_j \tilde{\alpha}_i \geq 0 \tag{3.2}$$

From this condition, we infer that *at most* one α_i is negative among all i, and at most one $\tilde{\alpha}_j$ is negative among all j. This condition does allow an arbitrary number of α_i or $\tilde{\alpha}_j$ to be zero. There are three possibilities for a given factor G_i:

1. **Type 0**: One of $\alpha_i, \tilde{\alpha}_i$ is zero.

2. **Type N**: One of $\alpha_i, \tilde{\alpha}_i$ is negative.

3. **Type P**: Both $\alpha_i, \tilde{\alpha}_i$ are positive.

For any model in the family, there can be at most two type N factors.

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We first show that the number of type 0 factors in the gauge group is unbounded for any infinite family. Assume the contrary is true, that is the number of such factors is bounded above by K over the entire family. There are at most two type N factors, so at least $N-K-2$ factors are of type P. For every pair of groups G_i and G_j among these $N-K-2$ factors, $\alpha_i\tilde{\alpha}_j + \alpha_j\tilde{\alpha}_i$ is strictly positive, so there are matter hypermultiplets charged simultaneously under $G_i \times G_j$. If each of these matter hypermultiplets are charged under at most two groups then the number of such hypermultiplets is $(N - K - 2)(N - K - 3)/2$, which goes as $\mathcal{O}(N^2)$ for large N. This is an overcount of the number of hypermultiplets needed, however, because there could be a single hypermultiplet charged under $\lambda > 2$ factors, which would be overcounted $\frac{\lambda(\lambda-1)}{2}$ times. Let λ denote the maximum number of gauge group factors that any hypermultiplet transforms under. The dimension of such a representation $\geq 2^\lambda$. The number of vector multiplets scales linearly with N, since each factor in the gauge group is bounded in dimension. As a result, $\lambda \sim \mathcal{O}(\log N)$; if it scales faster with N, the gravitional anomaly condition $n_h - n_v = 244$ would be violated. In the worst case, if all the matter hypermultiplets transform under $\lambda \sim \log N$ gauge group factors, n_h still grows as $\mathcal{O}(N^2/\log N)$. Therefore, $n_h - n_v \sim \mathcal{O}(N^2/\log N)$ as $N \to \infty$. This would clearly violate the gravitational anomaly condition (2.8) at sufficiently large N.

The above argument shows that the number of type 0 factors with α or $\tilde{\alpha}$ equal to zero must be unbounded. It also shows that the number of type P factors with both $\alpha, \tilde{\alpha}$ strictly positive can grow at most as fast as $\mathcal{O}(\sqrt{N})$ (dropping factors of $\log N$). Therefore, there are $\mathcal{O}(N)$ factors of type 0. For each factor G_i of this type, the coefficient of the $(\mathrm{tr}F_i^2)^2$ term is zero. This implies that the third term in the anomaly polynomial (2.10) vanishes for each type 0 factor G_i, so

$$\sum_R x_R \mathrm{tr}_R F_i^4 = \mathrm{Tr}F_i^4 \tag{3.3}$$

where x_R is the number of hypermultiplets in representation R of the group G_i.

We now show that type 0 factors all give a positive contribution to $n_h - n_v$ in the gravitational anomaly condition (2.8), so that the number of these factors is bounded.

Claim 3.1. *Every gauge group factor G_i that satisfies (3.3) also satisfies $h_i - v_i > 0$, where $h_i = \sum_R x_R \dim(R)$ denotes the number of hypermultiplets charged under G_i, and $v_i = \dim(G_i)$.*

The proof of this claim is in appendix A. Using the statement of the claim, we wish to show that $n_h - n_v$ is positive and grows without bound over the family. We must be careful since there could be hypermultiplets charged under multiple groups, which will be overcounted if we simply add $h_i - v_i$ for each factor G_i. It is easily checked that there is no matter charged simultaneously under three or more type 0 factors[3] For two factors G_i, G_j, hypermultiplets can be charged under $G_i \times G_j$ if $(\alpha_i, \tilde{\alpha}_i) = (+, 0)$ and $(\alpha_j, \tilde{\alpha}_j) = (0, +)$ (or the other way around). For large enough N, however, all the type 0 factors must have

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[3]If there was such a hypermultiplet charged under $G_1 \times G_2 \times \cdots \times G_n$, for $n > 2$, then for all pairs $x_{ij} \propto \alpha_i\tilde{\alpha}_j + \alpha_j\tilde{\alpha}_i \neq 0$. This is impossible, since for at least 2 factors, $\tilde{\alpha}_i = \tilde{\alpha}_j = 0$ or $\alpha_i = \alpha_j = 0 \Rightarrow x_{ij} = 0$ for every such pair.

either $\alpha_i = 0$ or $\tilde{\alpha}_i = 0$, and hence, there cannot be *any* hypermultiplet charged under two type 0 factors G_i and G_j. To prove this assertion, assume there is one factor G_1 of the form $(0, +)$ and $\mathcal{O}(N)$ factors are of the other type $(+, 0)$. This implies that the number of hypermultiplets charged under some representation R of G_1 is large and scales as $\mathcal{O}(N)$. This is impossible, since this factor must satisfy $\sum_R x_R \text{tr}_R F_1^4 = \text{Tr} F_1^4$, while the r.h.s. is fixed and $\text{tr}_R F_1^4 > 0$ for every R. Thus, for large enough N,

$$n_h - n_v = \sum_{\text{type } 0} (h_i - v_i) + \text{P/N-type contribution} \tag{3.4}$$

Since the dimension of each gauge group factor is bounded from above by M, the contribution of each factor is bounded from below $n_h - n_v \geq -M$. So, even if the $\mathcal{O}(\sqrt{N})$ type P factors contributed negatively,

$$n_h - n_v \geq \mathcal{O}(N) - M\mathcal{O}(\sqrt{N}) \tag{3.5}$$

$n_h - n_v$ is positive and grows as $\mathcal{O}(N)$, in contradiction with the gravitional anomaly condition (2.8).

This shows that given a finite list of simple groups, there is no infinite family of anomaly-free models, each of which has a gauge group consisting of an arbitrary product of simple groups in the list and matter in an arbitrary representation.

Case 2: Unbounded simple group factor. The only other way in which a gauge group could grow unbounded over a family of models is if the gauge group contains a classical group $H(N)$ (either $\text{SU}(N)$, $\text{SO}(N)$ or $\text{Sp}(N/2)$), whose rank grows without bound. Any such infinite family $\mathcal{F}$ necessarily contains an infinite sub-family of models with gauge group $H(N) \times G_N$ with $N \to \infty$. We now show that this case also leads to a contradiction.

Brief outline of the proof for case 2: We examine the $\text{tr}F^4$ conditions for models in the family with $N > \tilde{N}$. This allows us to show that every infinite family with an unbounded factor, at large enough N, has an infinite sub-family with gauge group and matter content parameterized in one of a few different ways. For each of these possible parameterizations of gauge group and associated matter fields, the contribution to $n_h - n_v$ diverges with N. This would violate the gravitational anomaly condition, unless there is a sufficiently negative contribution from the rest of the gauge group G_N and the associated charged matter. This, however, requires $\dim(G_N)$ to grow without bound in such a way that the contribution to $n_h - n_v$ is negative and divergent, which we show is impossible.

Consider first the case when the gauge group is of the form $\text{SU}(N) \times G_N$, where G_N is an arbitrary group. Anomaly cancellation (2.9) for the $\text{SU}(N)$ gauge group component requires that

$$B_{\text{Adj}} = 2N = \sum_R x_R B_R \tag{3.6}$$

where R is a representation of $\text{SU}(N)$. x_R denotes the total number of hypermultiplets that transform under representation R of $\text{SU}(N)$; it also includes hypermultiplets that are

Group	Representation	Dimension	A_R	B_R	C_R
	$\square$	N	1	1	0
	Adjoint	N^2-1	$2N$	$2N$	6
	$\square\!\!\square$ (antisym.)	$\frac{N(N-1)}{2}$	$N-2$	$N-8$	3
SU(N)	$\square\square$ (sym.)	$\frac{N(N+1)}{2}$	$N+2$	$N+8$	3
	(3-index antisym.)	$\frac{N(N-1)(N-2)}{6}$	$\frac{N^2-5N+6}{2}$	$\frac{N^2-17N+54}{2}$	$3N-12$
	(mixed)	$\frac{N(N^2-1)}{3}$	N^2-3	N^2-27	$6N$
	(3-index sym.)	$\frac{N(N+1)(N+2)}{6}$	$\frac{N^2+5N+6}{2}$	$\frac{N^2+17N+54}{2}$	$3N+12$
	$\square$	N	1	1	0
SO(N), Sp($\frac{N}{2}$)	(antisym.)	$\frac{N(N-1)}{2}$	$N-2$	$N-8$	3
	(sym.)	$\frac{N(N+1)}{2}$	$N+2$	$N+8$	3

Table 2. Values of the group-theoretic coefficients A_R, B_R, C_R for some representations of SU(N), SO(N) and Sp($N/2$). Note that the adjoint representations of SO(N) and Sp($N/2$) are given by the 2-index antisymmetric and the 2-index symmetric representation respectively.

charged under both SU(N) and G_N. For example, in a model with gauge group SU(N) $\times$ SU(4) with matter content $1(N,1) + 1(1,6) + 2(N,4) + c.c$, the number of hypermultiplets in the N of SU(N) would be counted as $x_N = 1 + 2 \times 4 = 9$. The coefficients A_R, B_R, C_R for various representations of SU(N) are shown in table 2.

The values of B_R for all representations of SU(N) other than the fundamental, adjoint, and two-index symmetric and antisymmetric representations grow at least as fast as $\mathcal{O}(N^2)$ as $N \to \infty$. For any given infinite family, there thus exists an $\tilde{N}$ such that for all $N > \tilde{N}$, the models in the family have no matter in any representations other than these. This follows because the l.h.s. of equation (3.6) scales as $\mathcal{O}(N)$, and each $x_R \geq 0$ on the r.h.s. . Thus, at large enough N, we only need to consider the fundamental, adjoint, symmetric and anti-symmetric representations. For these representations, (3.6) reads

$$2N = x_1 + 2Nx_2 + (N-8)x_3 + (N+8)x_4 \tag{3.7}$$

The only solutions to this equation when N is large are shown in table 3. We have discarded solutions $(x_1, x_2, x_3, x_4) = (0,1,0,0)$ and $(0,0,1,1)$, where $\alpha = \tilde{\alpha} = 0$ so the kinetic term for the gauge field is identically zero. We repeat the same analysis for the groups SO(N) and Sp($N/2$). At large values of N, we only need to consider the fundamental, anti-symmetric and symmetric representations. The allowed matter hypermultiplets for models with non-vanishing kinetic term are shown in table 3.

We have shown that for any infinite family of models where the rank of one of the gauge group factors increases without bound, the matter content is restricted to one of those in table 3. Notice that for each of these possibilities the contribution to $n_h - n_v$ from matter charged under $H(N)$ diverges as $N \to \infty$ (either as $\mathcal{O}(N)$ or $\mathcal{O}(N^2)$). Since the gauge group is $H(N) \times G_N$, the contribution to $n_h - n_v$ from G_N and associated matter charged under this group must be negative and unbounded (at least $-\mathcal{O}(N)$ or $-\mathcal{O}(N^2)$) in order to satisfy the gravitational anomaly condition. If we assume that the dimension

Group	Matter content	$n_h - n_v$	$\alpha, \tilde{\alpha}$
SU(N)	$2N\ \square$	$N^2 + 1$	$2, -2$
	$(N+8)\ \square + 1\ \square\!\square$ (antisym)	$\frac{1}{2}N^2 + \frac{15}{2}N + 1$	$2, -1$
	$(N-8)\ \square + 1\ \square\square$ (sym)	$\frac{1}{2}N^2 - \frac{15}{2}N + 1$	$-2, 1$
	$16\ \square + 2\ \square\!\square$ (antisym)	$15N + 1$	$2, 0$
SO(N)	$(N-8)\ \square$	$\frac{1}{2}N^2 - \frac{7}{2}N$	$-2, 1$
Sp($N/2$)	$(N+8)\ \square$	$\frac{1}{2}N^2 + \frac{7}{2}N$	$2, -1$
	$16\ \square + 1\ \square\!\square$ (antisym)	$15N - 1$	$2, 0$

Table 3. Allowed charged matter for an infinite family of models with gauge group $H(N)$. The last column gives the values of $\alpha, \tilde{\alpha}$ in the factorized anomaly polynomial.

of each simple factor in G_N is bounded, and that the number of simple factors diverges, we arrive at a contradiction in exactly the same way as we did in Case 1 above: $n_h - n_v$ is positive and scales at least linearly with the number of factors. G_N must therefore contain another classical group factor $\hat{H}(M)$, whose dimension also increases without bound over the infinite family.

Therefore, any given infinite family must have an infinite sub-family, with gauge group of the form $\hat{H}(M) \times H(N) \times \tilde{G}_{M,N}$, with both $M, N \to \infty$. Note that values of $(\alpha, \tilde{\alpha})$ for each unbounded classical factor are restricted to the values $(\pm 2, \mp 2)$, $(\pm 2, \mp 1)$, $(\pm 1, \mp 2)$, $(2, 0)$, $(0, 2)$. If F_1 denotes the field strength of the $\hat{H}(M)$ factor and F_2 that of $H(N)$, the coefficient of the $\text{tr} F_1^2 \text{tr} F_2^2$ term in the anomaly polynomial is $4 \sum_{R,S} A_R A_S x_{RS}$, where x_{RS} is the number of hypermultiplets in the (R, S) representation of $\hat{H}(M) \times H(N)$. Comparing coefficients, we have

$$\alpha_1 \tilde{\alpha}_2 + \alpha_2 \tilde{\alpha}_1 = 4 \sum_{R,S} A_R A_S\, x_{RS} \in 4\mathbb{Z} \tag{3.8}$$

From table 3, the coefficients $\alpha_i, \tilde{\alpha}_i$ do not diverge with N and M, and therefore the l.h.s. also does not diverge. The group-theoretic factors A_R and A_S on the other hand, have positive leading terms divergent with N and M for all representations except the fundamental. Hence, R and S are restricted to be the fundamental representations of the groups $\hat{H}(M)$ and $H(N)$ respectively. The r.h.s. of the above equation is, then, just four times the number of bi-fundamentals and therefore non-negative. Since at most one α_i, and at most one $\tilde{\alpha}_i$ can be negative, we can without loss of generality fix $\alpha_1 > 0$ and $\alpha_1 \geq \tilde{\alpha}_2$. If we require that $\alpha_1 \tilde{\alpha}_2 + \alpha_2 \tilde{\alpha}_1$ be divisible by four and non-negative, the possible values for $(\alpha_1, \tilde{\alpha}_1, \alpha_2, \tilde{\alpha}_2)$ are:

1. $(2, -2, 0, 2), (2, -1, 0, 2), (2, 0, 0, 2)$: For each of these values, there is one bi-fundamental hypermultiplet charged under $\hat{H}(M) \times H(N)$. As a consequence, the number of hypermultiplets charged under the fundamental representation of $H(N)$ scales linearly with M, which diverges. This is in contradiction with table 3 where the number of hypermultiplets in the fundamental is fixed at 16.

2. $(2, 0, 2, 0)$: In this case, the number of bi-fundamentals is zero. The contribution to $n_h - n_v$ from $\hat{H}(M) \times H(N)$ diverges with M, N. If such an infinite family were to

Gauge Group	Matter content	Anomaly polynomial
$\mathrm{SU}(N) \times \mathrm{SU}(N)$	$2(\square, \bar{\square})$	$(X - 2Y + 2Z)(X + 2Y - 2Z)$
$\mathrm{SO}(2N + 8) \times \mathrm{Sp}(N)$	$(\square, \square)$	$(X - Y + Z)(X + 2Y - 2Z)$
$\mathrm{SU}(N) \times \mathrm{SO}(N + 8)$	$(\square, \square) + (\boxminus, 1)$	$(X + Y - Z)(X - 2Y + 2Z)$
$\mathrm{SU}(N) \times \mathrm{SU}(N + 8)$	$(\square, \square) + (\boxminus, 1) + (1, \square\square)$	$(X - 2Y + 2Z)(X + Y - Z)$
$\mathrm{Sp}(N) \times \mathrm{SU}(2N + 8)$	$(\square, \square) + (1, \square\square)$	$(X - 2Y + 2Z)(X + Y - Z)$

Table 4. Infinite families of anomaly-free 6D models, where the anomaly polynomial factorizes as shown. X, Y, Z denote $\mathrm{tr}R^2, \mathrm{tr}F_1^2, \mathrm{tr}F_2^2$ respectively, where F_1 is the field strength of the first gauge group factor and F_2 that of the second. In each of the above models, the number of neutral hypermultiplets is determined from the $n_h - n_v = 244$ condition. Note that the $\square$ of $\mathrm{SU}(N)$ can be exchanged for the $\bar{\square}$, to generate a different model.

exist, the same argument as before implies that $\tilde{G}_{M,N}$ must contain a classical group factor that grows without bound and cancels the divergent contribution to $n_h - n_v$. As this analysis shows, this factor must have $(\alpha, \tilde{\alpha}) = (2, 0)$, and we have learned that such factors contribute a positive, divergent amount to $n_h - n_v$. Therefore, there is no way that the divergent contribution from $\hat{H}(M) \times H(N)$ can be compensated by a negative divergence from factors in $\tilde{G}_{M,N}$ and so we have a contradiction in this case as well.

3. $(2, -2, -2, 2), (2, -1, -2, 1)$: These possibilities give rise to five infinite families of models where the anomaly factorizes and can be cancelled by the Green-Schwarz mechanism; these families are tabulated in table 3. In all the models in these families, we have $\alpha_1 = -\alpha_2, \tilde{\alpha}_1 = -\tilde{\alpha}_2$. Thus, at any given value of the dilaton, the sign of the gauge kinetic terms is opposite for the two gauge group factors, and most be negative for one gauge group. Thus, these models all have a perturbative instability. While it is possible that there is some way of understanding these tachyonic theories, they do not satisfy our condition of positive gauge kinetic terms, so we discard them. Note that the first two of these five infinite families were discovered by Schwarz [23], and the third was found in [17]. The argument presented here shows that this list of five families is comprehensive.

Note that there are no families with three or more gauge groups of unbounded rank satisfying anomaly factorization. The analysis above gives the values allowed of the $\alpha, \tilde{\alpha}$'s for each possible pair of groups. With three groups, additional bifundamental matter fields between each pair would be needed so that the matter content for any component group could not be among the possibilities listed in table 3.

This completes the proof of Case 2, showing that there are no infinite families containing a simple group factor of unbounded rank. This in turn completes the proof that there exists no infinite family of models with anomalies that can be cancelled by the Green-Schwarz mechanism and with kinetic terms positive in some region of moduli space. □

4 Classification of models

In proving the finiteness of the number of possible gauge fields and matter representations which are possible in chiral 6D SUSY gauge theories, we have developed tools which could lead directly to a systematic enumeration of all possible consistent low-energy models of this type. A previous enumeration of some of these models with one and two gauge group factors was carried out in [24].

In particular, it is possible to use the gravitational anomaly and anomaly factorization conditions to analyze possible gauge groups in a systematic way by considering each simple factor of the gauge group separately. The anomaly conditions (2.9), (2.11) and (2.12) constrain the possible sets of matter fields which transform under any given simple factor in the gauge group, independent of what other factors appear in the full gauge group. The gravitational anomaly (2.8) places a strong limit on the number of hypermultiplet matter fields which can be included in the theory. Since, as we found in several places in the proof in the preceding section, the contribution of components of the gauge group to $n_h - n_v$ is generally positive, we can think of the problem of enumerating all possible gauge and matter configurations for chiral 6D supergravity theories as like a kind of partition problem. This problem is complicated by the matter fields charged under more than one gauge group, which contribute to n_h only once. Nonetheless, by analyzing individual simple factors and associated allowed matter representations, we can determine a set of building blocks from which all chiral 6D theories may be constructed. The contribution to $n_h - n_v$ is reasonably large for most possible blocks (it seems that only a small number of gauge group factors and associated matter configurations give negative contributions [which can only appear once] or positive contributions of much less than 30 or 40 to $n_h - n_v$). Furthermore, the number of bifundamental fields grows as the square of the number of type P factors in the gauge group, so it seems that the combinatorial possibilities for combining blocks are not too vast. A very crude estimate suggests that the total number of models may be under a billion, and that a complete tabulation of all consistent models is probably possible. Certainly we do not expect anything like the $\sim 10^{500}$ distinct models which can arise from 4D type IIB flux compactifications. It is also worth mentioning that as n_t increases, the allowed contribution to $n_h - n_v$ decreases, so that the total number of possibilities may decrease rapidly for larger n_t despite the more complicated anomaly-cancellation mechanism. We will give a more systematic discussion of the block-based construction of models in a future paper.

As an example of the limitations on this type of "building blocks", consider $\mathrm{SU}(N)$ with x_f hypermultiplet fields in the fundamental representation and x_a in the antisymmetric two-index representation. The condition (2.9) relates x_a and x_f through $x_f = 2N - x_a(N-8)$; this relation allows us to write the contribution to the gravitational anomaly as $n_h - n_v = 1 + N(x_f + 7x_a)/2$. This quantity is necessarily positive, and can be used to bound the allowed values of $N, x_f,$ and x_a either for a group with one factor $\mathrm{SU}(N)$ or with several factors of which one is $\mathrm{SU}(N)$. A similar analysis can be carried out for larger representations of $\mathrm{SU}(N)$. For example, if we allow the 3-index antisymmetric representation $\boxed{}$, with a gauge group $\mathrm{U}(N)$ and no additional factors, we find that no

model has matter in the representation ☐ unless $N \leq 8$. There are some examples of this type of model, such as the SU(7) theory with matter $24\,\square + 2\,\boxed{}\, + \boxed{}$, which satisfy anomaly cancellation and which are (we believe) not yet identified as string compactifications.[4]

A complete classification of allowed 6D theories would have a number of potential applications. In analogy to the story in 10D, where such a classification led to the discovery of the heterotic $E_8 \times E_8$ string, discovery of novel consistent low-energy models in six dimensions may suggest new mechanisms of string compactification. Or, finding a set of apparently consistent theories which do not have string realizations might lead either to a discovery of new consistency conditions which need to be imposed on the low-energy theory, or to a clearer understanding of a 6D "swampland" [25, 26] of apparently consistent theories not realized in string theory. If all theories satisfying the constraints we are using here can be either definitively realized as string compactifications or shown to be inconsistent, it would prove the conjecture of "string universality" stated in [17] for chiral 6D supersymmetric theories, at least for the class of models with one tensor multiplet and no U(1) gauge factors.

The proof given here has shown that there are a finite number of distinct gauge groups and matter content which can be realized in chiral 6D supergravity theories with one tensor multiplet. We have not, however, addressed the question of whether a given gauge group and matter content can be associated with more than one UV-complete supergravity theory (by "theory" here, meaning really a continuous component of moduli space). In [27], we showed that for one class of gauge groups, almost all anomaly-free matter configurations are realized in a unique fashion in heterotic compactifications, but that some models can be realized in distinct fashions characterized by topological invariants described in that case by the structure of a lattice embedding. It would be interesting to understand more generally the extent to which the models considered here have unique UV completions.

A complete classification of allowed theories in six dimensions could lead to a better understanding of how various classes of string compactifications populate the string landscape. Such lessons might be helpful in understanding the more complicated case of four-dimensional field theories with gravity. As mentioned in section 2, in some situations the anomaly constraints in six dimensions correspond precisely to the constraints on string compactifications; in both cases these constraints follow from various index theorems. Making this connection more precise in six dimensions could shed new light on the relationship between string theory and low-energy effective theories in any dimension.

5   Conclusions and future directions

We have shown that for 6D $(1, 0)$ supersymmetric theories with gravity, one tensor multiplet, a semi-simple gauge group and hypermultiplet matter in an arbitrary representation,

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[4]As this work was being completed we learned that Grassi and Morrison have found a local construction of this kind of 3-index antisymmetric representation through string compactification in the language of F-theory [22]

the conditions from anomaly cancellation and positivity of the kinetic terms suffice to prove finiteness of the set of possible gauge groups and matter content.

In this analysis we have only considered semi-simple gauge groups. When there are U(1) factors in the gauge group, there is a generalized Green-Schwarz mechanism discussed in [28], which involves the tree-level exchange of a 0-form. Addressing the question of finiteness of theories including U(1) factors would require further analysis.

We have also restricted attention here to theories with one tensor multiplet, which admit a low-energy Lagrangian description [29]. There are many string compactifications which give rise to 6D models with more than one tensor multiplet [30–34]. While a proof of finiteness for models with more tensor multiplets is probably possible, the exchange of multiple anti-self-dual tensor fields in the Green-Schwarz mechanism as described by Sagnotti [9] makes this analysis more complicated, and we leave a treatment of such cases to future work.

A comment may also be helpful on non-chiral theories with $(1, 1)$ supersymmetry; this issue is addressed in further detail in appendix B. A $(1, 1)$ model in six dimensions also has $(1, 0)$ supersymmetry, but contains an additional $(1, 0)$ gravitino multiplet beyond the gravity, tensor, hyper and vector $(1, 0)$ multiplets in the theories we have considered here. It seems that one cannot include the gravitino multiplet without having $(1, 1)$ local supersymmetry, and we have restricted our attention here to models with $(1, 0)$ supersymmetry without this gravitino multiplet. Since $(1, 1)$ models are non-chiral, they cannot be constrained by anomaly cancellation. Some further mechanism would be needed to constrain the set of $(1, 1)$ supersymmetric models in six dimensions. It is possible that such constraints could be found by demonstrating that string-like excitations of the theory charged under the tensor field must be included in the quantum theory; anomalies in the world-volume theory of these strings would then impose constraints on the 6D bulk theory, as suggested in [20]. It may also be that understanding the dictionary between anomaly constraints and constraints arising from string compactification for chiral theories may suggest a new set of constraints even for non-chiral 6D theories.

The result of this paper ties into the question of the number of topologically distinct Calabi-Yau manifolds, since 6D supergravity theories can be realized by compactification of F-theory on elliptically fibered Calabi-Yau manifolds [35]. It has been shown by Gross [36] that there are only finitely many Calabi-Yau manifolds that admit an elliptic fibration, up to birational equivalence. If we can prove that the set of $(1, 0)$ models with any number of tensor multiplets is finite then this would constitute a "physics proof" of the theorem.

The total number of consistent 6D models of the type we consider does not seem to be enormous. It is not hard to imagine that these theories could be completely enumerated in a systematic manner. This programme would be very useful to understand the structure of the landscape and swampland [25] in this special case of six dimensions with $(1, 0)$ supersymmetry. In analogy with the discovery of the $E_8 \times E_8$ heterotic string, we are optimistic that further study of the set of consistent 6D supergravity theories will help us better understand the rich structure of string compactifications, perhaps giving lessons which will be relevant to the more challenging case of compactifications to four dimensions.

Acknowledgments

A Proof of claim 3.1

In the proof of finiteness in section 3, we claimed that for any group G if

$$\sum_R x_R \text{tr}_R F^4 = \text{Tr} F^4, \tag{A.1}$$

for $x_R \in \mathbb{Z}, x_R \geq 0$ and R runs over all the representations of G, then

$$h - v = \sum_R x_R \dim(R) - \dim(G) \geq c > 0 \tag{A.2}$$

Equation (A.1) is automatically satisfied if $x_R = 1$ for the adjoint representation and 0 for all other representations. In this case, however, the kinetic terms for the gauge group factor G would be zero, and we are not considering this case (See appendix B for a discussion of this situation in the context of $(1,1)$ supersymmetry). If $x_R > 0$ for a representation R with $\dim(R) > \dim(G)$, equation (A.2) is automatically satisfied. We only need to consider situations where $x_R = 0$ for all representations R with $\dim(R) > \dim(G)$.

A.1 Exceptional groups

We first consider the exceptional groups G_2, F_4, E_6, E_7, E_8. The only representation that satisfies $\dim(R) \leq \dim(G)$ in all these cases is the fundamental. Equation (A.1) is satisfied for these groups, if the number of hypermultiplets are:

- G_2: $x_f = 10 \Rightarrow h - v = 70 - 14 = 56 > 0$

- F_4: $x_f = 5 \Rightarrow h - v = 130 - 52 = 78 > 0$

- E_6: $x_f = 6 \Rightarrow h - v = 162 - 78 = 84 > 0$

- E_7: $x_f = 4 \Rightarrow h - v = 224 - 133 = 91 > 0$

- E_8: There are no solutions because the fundamental representation of E_8 is the adjoint representation.

A.2 SU(N)

We first consider the case when $N \geq 4$, and then the cases SU(2) and SU(3). The set of irreducible representations are in one-one correspondence with the set of Young diagrams with up to $N - 1$ rows. Equation (A.1) implies that

$$\sum_R x_R C_R = 6 \tag{A.3}$$

Since $x_R \geq 0$, we only need to consider representations which have $C_R \leq 6$, where the coefficients C_R are defined in (2.6). For a given representation R of $SU(N)$, choose $F = F_{12}T_R^{12} + F_{34}T_R^{34}$. The generators T^{12} and T^{34} are given (in the fundamental representation) by

$$(T^{12})_{ab} = \delta_{a1}\delta_{b1} - \delta_{a2}\delta_{b2} \tag{A.4}$$

$$(T^{34})_{ab} = \delta_{a3}\delta_{b3} - \delta_{a4}\delta_{b4}, \ a,b = 1,2,\ldots,N \tag{A.5}$$

Substituting this form of F into the definition (2.6) for C_R, we have

$$\mathrm{tr}_R(F_{12}T_R^{12} + F_{34}T_R^{34})^4 = (2B_R + 4C_R)(F_{12}^4 + F_{34}^4) + 8C_R F_{12}^2 F_{34}^2 \tag{A.6}$$

Comparing coefficients of $F_{12}^2 F_{34}^2$ on both sides, we have the following formula for C_R:

$$C_R = \frac{3}{4}\mathrm{tr}_R[(T_R^{12})^2(T_R^{34})^2] \tag{A.7}$$

Using the above formular for C_R, we can show that the only representations of $SU(N)$ that satisfy $C_R \leq 6$ are

$$\text{Adjoint,} \ \square, \ \square\square, \ \begin{array}{c}\square\\\square\end{array} \text{ for all } N, \text{ and } \begin{array}{c}\square\\\square\\\square\end{array} (C_R = 6 \text{ for } N = 6) \tag{A.8}$$

Examples.

1. $\square\square$: We must compute the trace in (A.7). The only states that give a non-zero contribution are

$$\boxed{1\,3}, \ \boxed{2\,3}, \ \boxed{1\,4}, \ \boxed{2\,4}$$

   This gives $C = 3$ for the two-index symmetric representation.

2. $\begin{array}{c}\square\\\square\end{array}$: The only states that contribute to the trace are:

$$\begin{array}{c}\boxed{1}\\\boxed{3}\end{array}, \ \begin{array}{c}\boxed{2}\\\boxed{3}\end{array}, \ \begin{array}{c}\boxed{1}\\\boxed{4}\end{array}, \ \begin{array}{c}\boxed{2}\\\boxed{4}\end{array} \tag{A.9}$$

   Again, here $C = 3$

3. $\begin{array}{c}\square\square\\\square\end{array}$: The following states contribute to the trace:

$$\begin{array}{cc}\boxed{1\,3}\\\boxed{i}\end{array}, \ \begin{array}{cc}\boxed{2\,4}\\\boxed{i}\end{array}, \ \begin{array}{cc}\boxed{1\,4}\\\boxed{i}\end{array}, \ \begin{array}{cc}\boxed{2\,3}\\\boxed{i}\end{array},$$

$$\begin{array}{cc}\boxed{1\,i}\\\boxed{3}\end{array}, \ \begin{array}{cc}\boxed{2\,i}\\\boxed{4}\end{array}, \ \begin{array}{cc}\boxed{1\,i}\\\boxed{4}\end{array}, \ \begin{array}{cc}\boxed{2\,i}\\\boxed{3}\end{array},$$

$$\begin{array}{cc}\boxed{1\,1}\\\boxed{3}\end{array}, \ \begin{array}{cc}\boxed{1\,1}\\\boxed{4}\end{array}, \begin{array}{cc}\boxed{2\,2}\\\boxed{3}\end{array}, \ \begin{array}{cc}\boxed{2\,2}\\\boxed{4}\end{array}, \ \begin{array}{cc}\boxed{3\,3}\\\boxed{1}\end{array}, \ \begin{array}{cc}\boxed{3\,3}\\\boxed{2}\end{array}, \ \begin{array}{cc}\boxed{4\,4}\\\boxed{1}\end{array}, \ \begin{array}{cc}\boxed{4\,4}\\\boxed{2}\end{array}$$

   Here i denotes any of the remaining $N - 4$ indices. For this representation

$$C = \frac{3}{4}\left(4(N-4) + 4(N-4) + 4 \times 8\right) = 6N \tag{A.10}$$

Any representation of SU(N) can be represented as a Young diagram with at most $N-1$ rows. For a general Young diagram, which does not correspond to the representations listed in (A.8), we can show that $C_R > 6$, by explicitly enumerating states that contribute to C_R as in the examples above. In proving this generally, it is useful to note that adding boxes to any nonempty horizontal row only increases the value of C, so it is sufficient to consider only the totally antisymmetric representations and case 3 above.

The only solutions to (A.1) are:

1. $1\,\square\!\square + 1\,\square\square\!\square$: $h - v = 1 > 0$. In this case $\alpha = \tilde{\alpha} = 0$, so we discard this solution.

2. $16\,\square + 2\,\square\!\square$: $h - v = 15N + 1 > 0$

3. $1\,\square\!\square + 24\,\square$, $(N = 6)$: $h - v = 20 + 144 - 35 = 129 > 0$

Now consider the case of SU(2). Since $B_R = 0$, (A.1) for SU(2) becomes

$$\sum_R x_R C_R = 8 \tag{A.11}$$

We therefore must enumerate all representations where $C_R \leq 8$. The formula for C_R is simpler in the SU(2) case:

$$C_R = \frac{1}{4}\mathrm{tr}_R\left[\left(T_R^{12}\right)^4\right] \tag{A.12}$$

Young diagrams for $SU(2)$ only have one row. For a diagram with c columns, the states $\boxed{1\,1\,\cdots\,1}$ and $\boxed{2\,2\,\cdots\,2}$ show that $C_R \geq c^4/2 > 8$ for $c > 2$. For $c = 2$, we have the adjoint representation with $C_R = 8$, which would lead to $\alpha = \tilde{\alpha} = 0$. The only possible remaining solution to (A.11) is $16\,\square$ hypermultiplets, which has $h - v = 29 > 0$.

For SU(3), (A.1) becomes

$$\sum_R x_R C_R = 9 \tag{A.13}$$

We can compute C_R using formula (A.12) for the SU(3) case as well. The Young diagrams for SU(3) contain at most two rows. For a diagram with c columns in the first row, the following states

$$\begin{array}{c}\boxed{\begin{array}{ccc}1&1&\cdots\,1\\3&\cdots\,3&\end{array}}\quad\boxed{\begin{array}{ccc}2&2&\cdots\,2\\3&\cdots\,3&\end{array}}\end{array} \tag{A.14}$$

give a lower bound $C_R \geq c^4/2$, with $C_R > 9$ for $c > 2$. The only representations (besides the adjoint) with $C_R \leq 9$ are $\square$, $\square\!\square$ with $C_R = 1/2$, $17/2$. The only combination of matter representations that satisfies (A.13) with one of $\alpha, \tilde{\alpha}$ positive is $18\,\square$ with $h - v = 46 > 0$.

A.3 SO(N)

For $N \leq 6$, these are related to other simple Lie groups, so we only consider $N \geq 7$. We choose the commuting generators T^{12} and T^{34} of SO(N), defined as

$$(T^{12})_{ab} = i\delta_{a2}\delta_{b1} - i\delta_{a1}\delta_{b2} \tag{A.15}$$

$$(T^{34})_{ab} = i\delta_{a4}\delta_{b3} - i\delta_{a3}\delta_{b4} \tag{A.16}$$

N	Matter	$h - v$
7	8 spinor $+ 3 \square$	$64 + 21 - 21 = 64 > 0$
8	8 spinor $+ 4 \square$	$64 + 32 - 28 = 68 > 0$
9	4 spinor $+ 5 \square$	$64 + 45 - 36 = 73 > 0$
10	4 spinor $+ 6 \square$	$64 + 60 - 45 = 79 > 0$
11	2 spinor $+ 7 \square$	$64 + 77 - 55 = 86 > 0$
12	2 spinor $+ 8 \square$	$64 + 96 - 66 = 94 > 0$
13	1 spinor $+ 9 \square$	$64 + 117 - 78 = 113 > 0$
14	1 spinor $+ 10 \square$	$64 + 126 - 91 = 99 > 0$

Table 5. Matter hypermultiplets that solve (A.1) for SO(N), $7 \leq N \leq 14$. All solutions for $N > 14$ have positive $h - v$.

Notice that the squared SO(N) generators are identical to the squared SU(N) generators we used in the previous section. Thus, the formulae for any Young diagram carry over. In the case of SO(N), the Young diagrams are restricted so that the total number of boxes in the first two columns does not exceed N [37]. The antisymmetric representation is the adjoint, and so we need to find all diagrams with $C_R \leq 3$. The only diagrams that satisfy this requirement are — fundamental, 2-index antisymmetric and the 2-index symmetric representation.

In addition to these, we also have the spinor representations of SO(N). The irreducible spinor representation of SO(N) has dimension $2^{\lfloor (N-1)/2 \rfloor}$. It is smaller than the adjoint only for $N \leq 14$, and it can be checked (using the tables in [38]) that the only spinor representation that is smaller in dimension than the adjoint is the Weyl/Dirac spinor. The trace formula for these is [14]

$$\mathrm{tr}_s F^2 = 2^{\lfloor (N+1)/2 \rfloor - 4} \mathrm{tr} F^2 \tag{A.17}$$

$$\mathrm{tr}_s F^4 = -2^{\lfloor (N+1)/2 \rfloor - 5} \mathrm{tr} F^4 + 3 \cdot 2^{\lfloor (N+1)/2 \rfloor - 7} (\mathrm{tr} F^2)^2 \tag{A.18}$$

For the spinor representation $C \leq 3$ for $7 \leq N \leq 14$. For each of these cases, we can solve for representations that solve (A.1), and check whether $h - v$ is positive. This is the case for all the solutions, and these are shown in table 5.

A.4 Sp(N)

By Sp(N), we mean the group of $2N \times 2N$ matrices that preserve a non-degenerate, skew-symmetric bilinear form. We choose as generators T^{12}, T^{34} in the fundamental representation as

$$(T^{12})_{ab} = \delta_{a1} \delta_{b2} + \delta_{a2} \delta_{b1} \tag{A.19}$$

$$(T^{34})_{ab} = \delta_{a3} \delta_{b4} + \delta_{a4} \delta_{b3}, \quad a, b = 1, 2 \cdots 2N \tag{A.20}$$

Again, these generators have been chosen so that their squares are equal to the squares of the generators of SU($2N$). The Young diagrams for Sp(N) are similar to those of SU($2N$), except that only diagrams with less than or equal to N rows need to be considered [37]. The

adjoint of $\mathrm{Sp}(N)$ is the symmetric representation. The analysis for the $\mathrm{SU}(N)$ case carries through, except that we only need to consider representations with $C \leq 3$. The only representations with this property are — fundamental, 2-index antisymmetric (traceless w.r.t skew-symmetric form) and the 2-index symmetric. The only solution is $1\,\square + 16\,\square$, and for this solution, $h - v = 30N + 1 > 0$. This proves Claim 3.1 in section 3. $\qquad\square$

B (1, 1) supersymmetry and the case of one adjoint hypermultiplet

In this paper we have focused on $(1,0)$ theories with chiral matter content.[5] In this appendix we discuss how these theories differ from $(1,1)$ non-chiral 6D supergravity theories.

One can imagine a $(1,0)$ theory where the matter content consists of precisely one hypermultiplet in the adjoint representation for each factor in the gauge group. Together, these $(1,0)$ multiplets combine to form the $(1,1)$ vector multiplet. The field content of this multiplet is [13]

$$A_\mu + 4\phi + \psi_+ + \psi_-$$

Since this matter content is non-chiral, there are no gauge anomalies or mixed gravitational-gauge anomalies; there is still a purely gravitational anomaly, which is cancelled in the usual way. In the framework of the discussion here, (2.9), (2.11), and (2.12) are all satisfied with $\alpha_i = \tilde{\alpha}_i = 0$. Thus, the gauge kinetic terms vanish and we do not consider models of this type in the analysis here.

It may seem that this contradicts the straightforward observation that all $(1,1)$ supergravity theories can be thought of as $(1,0)$ gravity theories while the gauge fields in $(1,1)$ supergravity theories can have nonvanishing gauge kinetic terms. The point, however, is that the model described above with adjoint matter for each gauge group component does not have $(1,1)$ (local) supersymmetry. To see this, note that the $(1,1)$ gravity multiplet consists of

$$\begin{aligned}
&g_{\mu\nu} + B^+_{\mu\nu} + B^-_{\mu\nu} + 4A_\mu + \phi + \psi^+_\mu + \psi^-_\mu + \chi_+ + \chi_- \\
&= (g_{\mu\nu} + B^-_{\mu\nu} + \psi^-_\mu) + (B^+_{\mu\nu} + \phi + \chi^+) + (4A_\mu + \psi^+_\mu + \chi^-)
\end{aligned}$$

In the $(1,0)$ language, this corresponds to the gravity multiplet, tensor multiplet and an additional gravitino multiplet. The $(1,0)$ gravitino multiplet consists of four abelian vectors, two Weyl fermions and one gravitino, and is scarcely even mentioned in the vast literature on 6D models. The reason for this is that if we consider a $(1,0)$ model with one gravitino multiplet, it seems that the model must have $(1,1)$ supergravity. This is certainly the case in string theories, where the vertex operator for the supercharge is the same as that of the gravitino. More generally, the common lore [12] states that a massless, interacting spin-3/2 field must couple to a local conserved supercurrent. This fact is compatible with

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[5]Thanks to Ken Intriligator for discussions on this issue.

the anomaly conditions. The anomaly polynomial in the case of one gravitino multiplet is

$$I = -\frac{n_h - n_v}{5760}\text{tr}R^4 - \frac{n_h - n_v}{4608}(\text{tr}R^2)^2 - \frac{1}{96}\text{tr}R^2\sum_i\left[\text{Tr}F_i^2 - \sum_R x_R^i\text{tr}_R F_i^2\right]$$

$$+ \frac{1}{24}\left[\text{Tr}F_i^4 - \sum_R x_R^i\text{tr}_R F_i^4\right] - \frac{1}{4}\sum_{i,j,R,S} x_{RS}^{ij}(\text{tr}_R F_i^2)(\text{tr}_S F_j^2) \qquad \text{(B.1)}$$

The gravitational anomaly can be cancelled only if $n_h - n_v = 0$. One way to cancel the anomaly is to have a single hypermultiplet transforming in the adjoint of the gauge group $\mathcal{G}$. In this case, the matter content is that of a $(1,1)$ 6D model with gravity and one vector multiplet. It would be nice to have a simple proof directly from the anomaly cancellation conditions that this is the only way to have $n_h - n_v = 0$, which would amount to a proof that any $(1,0)$ theory with a gravitino multiplet would have $(1,1)$ supersymmetry. In the case of two gravitino multiplets, it seems that the only consistent solution is the non-chiral, maximal $(2,2)$ 6D supergravity.

We can now understand the apparent discrepancy alluded to above. A $(1,0)$ model with one hypermultiplet in the adjoint of the gauge group $\mathcal{G}$ has $\alpha = \tilde{\alpha} = 0$, which implies that the gauge kinetic terms are zero. However, in a $(1,1)$ model with a vector multiplet corresponding to $\mathcal{G}$, the gauge kinetic term is positive. (An easy way to see this is to consider a T^4 compactification of the heterotic string, which gives $(1,1)$ supergravity in 6D). This apparent inconsistency is due to the fact that in the case of the $(1,1)$ theory, $\alpha \neq 0, \tilde{\alpha} = 0$, and this is sufficient to ensure that the anomaly polynomial vanishes. The B-field has the usual Chern-Simons coupling that makes it transform under gauge transformations. However, there is no $B \wedge d\tilde{\omega}$ term; the theory is non-anomalous and therefore there is no need for a Green-Schwarz counterterm.

Thus, we see that the conditions we impose in this paper on the field content of the $(1,0)$ theory exclude the gravitino multiplet needed to complete the $(1,0)$ graviton multiplet to a $(1,1)$ graviton multiplet. As a result, the class of theories considered here does not include theories with $(1,1)$ supersymmetry; the $(1,1)$ theories are all anomaly free and would need to be constrained by some alternative mechanism as discussed in 5.

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